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The Space $BV[a,b]$

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In the following $\mathbb{F}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$.

1 Definition and some properties

Definition 1. A function $f : [a,b] \to \mathbb{F}$ is said to be of **bounded variation** if there exists $M_f > 0$ such that for every partition $P : a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$,

$$V(f,P) := \sum_{i=1}^{n} |f(t_i) - f(t_{i-1}| \le M_f.$$

The quantity $V(f,P)$ is called the **variation of** f **over** P, and

$$V(f) := \sup_P V(f,P)$$

is called the **total variation** of f. ♢

The set of functions of bounded variation on $[a,b]$ is denoted by $BV[a,b]$.

Observe the following:

- $f \in BV[a,b] \iff V(f) < \infty$.
- For $f \in BV[a,b]$,
$$V(f) = 0 \iff f \quad \text{is a contant function.}$$

Remark 2. If $\mathbb{F} = \mathbb{C}$ and $f \in BV[a,b] \cap C[a,b]$, then $V(f)$ can be thought of as **length** of the curve $t \mapsto f(t)$. ♢

THEOREM 3. *If $f, g \in BV[a,b]$ and $\alpha \in \mathbb{F}$, then $f+g, \alpha f \in BV[a,b]$ and*

$$V(f+g) \le V(f) + V(g), \quad V(\alpha f) = |\alpha| V(f).$$

Proof. Let $f, g \in BV[a,b]$ and $\alpha \in \mathbb{F}$. Then for every partition P of $[a,b]$, we have

$$V(f+g,P) \le V(f,P) + V(g,P) \le V(f) + V(g), \quad V(\alpha f, P) = |\alpha| V(f,P).$$

Taking supremum over all such partitions, we obtain

$$V(f+g) \le V(f) + V(g), \quad V(\alpha f) = |\alpha| V(f).$$

Hence, $f+g$, $\alpha f \in BV[a,b]$. □

THEOREM 4. *For every $f \in BV[a,b]$,*

$$\sup_{a \le t \le b} |f(t)| \le |f(a)| + V(f).$$

In particular,

$$BV[a,b] \subseteq B[a,b]$$

and

$$\|f\|_\infty \le |f(a)| + V(f) \quad \forall f \in BV[a,b].$$

Proof. Let $f \in BV[a,b]$. For $t \in (a,b)$, consider the partition $P = \{a, t, b\}$. Then we have

$$|f(x) - f(a)| \le V(f, P) \le V(f).$$

Hence,

$$|f(x)| \le |f(a)| + V(f)|.$$

Hence the results follow. □

THEOREM 5. *The set $BV[a,b]$ is a linear space and*

$$\|f\|_{BV} := |f(a)| + V(f)$$

defines a norm on $BV[a,b]$, and $BV[a,b]$ is a Banach space with respect to the norm $\|f\|_{BV}$.

Proof. Recall that for $f \in BV[a,b]$, $V(f) = 0$ if and only if f is a constant function. Hence,

$$\|f\|_{BV} = 0 \iff f = 0.$$

Now the facts that $BV[a,b]$ is a linear space and $\|\cdot\|_{BV}$ is a norm on it follow from Theorem 3 and Theorem 4.

It remains to show that $\|\cdot\|_{BV}$ is a complete norm. For this, let (f_n) be a Cauchy sequence with respect to $\|\cdot\|_{BV}$. So, for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(a) - f_m(a)| + V(f_n - f_m) < \varepsilon \quad \forall m, n \ge N. \tag{$*$}$$

Hence, by Theorem 4, (f_n) is a Cauchy sequence in $BV[a,b]$. Since $BV[a,b]$ is a Banach space with respect to $\|\cdot\|$, there exists $f \in B[a,b]$ such that
$f_n - f\|_\infty \to 0$ as $n \to \infty$. In particular, $f_n \to f$ pointwise. Therefore, for every partition P of $[a,b]$,

$$|f_n(a) - f(a)| + V(f_n - f_m, P) = \lim_{m\to\infty} \{|f_n(a) - f_m(a)| + V(f_n - f_m, P)\}.$$

Hence, by ($*$),

$$|f_n(a) - f(a)| + V(f_n - f_m, P) \le \varepsilon \quad \forall n \ge N.$$

Since this is true for every partition of $[a,b]$, it follows that $f_n - f \in BV[a,b]$, and in particular, $f = f_n - (f_n - f) \in BV[a,b]$, and

$$\|f_n - f\|_{BV} = |f_n(a) - f(a)| + V(f_n - f_m) \le \varepsilon \quad \forall n \ge N.$$

Thus, (f_n) converges to $f \in BV[a,b]$ with respect to $\|\cdot\|_{BV}$. □

THEOREM 6. *Let $f \in BV[a,b]$ be a real valued function. Then there exists a monotonically increasing function $g : [a,b] \to \mathbb{R}$ such that both g and $g - f$ are monotonically increasing. In particular, f is difference of two monotonically increasing functions.*

Proof. For $[a,d] \subseteq [a,b]$, let $V_c^d(f)$ be the total variation of f on $[a,d]$. Then it can be verified that for any $t \in [a,b]$,

$$V_a^b(f) = V_a^t(f) + V_t^b(f).$$

Hence, for $a \le t < s \le b$,

$$V_a^s(f) - V_a^t(f) = V_t^s(f) \ge |f(s) - f(t)|.$$

Hence, the function $g(t) := V_a^t(f)$ is monotonically increasing. We also have

$$g(s) - g(t) = V_a^s(f) - V_a^t(f) = V_t^s(f) \ge |f(s) - f(t)| \ge f(s) - f(t).$$

Hence

$$g(s) - f(s) \ge g(t) - f(t).$$

Therefore, $g - f$ is also monotonically increasing. Clearly, every monotonically increasing function belongs to $BV[a,b]$. Thus, $f = g - h$ with $h := g - f$, where g and h are monotonically increasing functions. □

For more results on functions of bounded variation, one may refer Chapter 13 in [1]

2 Riemann-Stieltjes integral

Recall from the theory of Riemann-Stieltjes integral:

Definition 7. A function $f : [a,b] \to \mathbb{R}$ is **Riemann-Stieltjes integrable** with respect to a a monotonically increasing function $v : [a,b] \to \mathbb{R}$, if there exists $\gamma \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ satisfying

$$|S(P, \Delta) - \gamma| < \varepsilon$$

whenever

$$|P| := \max_i (t_i - t_{i-1}) < \delta$$

for every partition $P : a = t_0 < t_1 < \ldots < t_k = b$ of $[a,b]$ and for a set $\Delta := \{\tau_j\}$ of *tags* on P, that is, with $\tau_j \in [t_{j-1}, t_j]$, $j = 1, \ldots, k$, where the *Riemann-Stieltjes sum* $S(P, \Delta)$ is defined by

$$S(P, \Delta) := \sum_{i=1}^{k} f(\tau_i)[v(t_i) - v(t_{i-1})],$$

and in that case the number γ is called the Riemann-Stieltjes integral of f with respect to v, and it is denoted by $\int_a^b f dv$. ♢

If f is Riemann-Stieltjes integrable, then we write

$$\int_a^b f dv = \lim_{|P|\to 0} S(P,\Delta).$$

The following theorem is known.

THEOREM 8. *Every real valued $f \in C[a,b]$ is Riemann-Stieltjes integrable* with respect to *any monotonically increasing function.*

Definition 9. A function $f : [a,b] \to \mathbb{F}$ is **Riemann-Stieltjes integrable** with respect to a function $\varphi \in BV[a,b]$, if there exists $\gamma \in \mathbb{F}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ satisfying

$$|S(P,\Delta) - \gamma| < \varepsilon$$

whenever

$$|P| := \max_i (t_i - t_{i-1}) < \delta$$

for every partition $P : a = t_0 < t_1 < \ldots < t_k = b$ of $[a,b]$ and for a set $\Delta := \{\tau_j\}$ of *tags* on P, where

$$S(P,\Delta) := \sum_{i=1}^{k} f(\tau_i)[\varphi(t_i) - \varphi(t_{i-1})],$$

and in that case the number γ is called the Riemann-Stieltjes integral of f with respect to φ, and it is denoted by $\int_a^b f d\varphi$. ♢

In view of Theorem 6 and Theorem 10, we have the following theorem.

THEOREM 10. *Every $\mathbb{F}$-valued $f \in C[a,b]$ is Riemann-Stieltjes integrable* with respect to *any $\varphi \in BV[a,b]$.*

If $f \in C[a,b]$ and $\varphi \in BV[a,b]$, then for every partition P on $[a,b]$ and for every set Δ of tags on P, we have

$$\left|\sum_{i=1}^{k} f(\tau_i)[\varphi(t_i) - \varphi(t_{i-1})]\right| \leq \sum_{i=1}^{k} |f(\tau_i)|\,|[\varphi(t_i) - \varphi(t_{i-1})]| \leq \|f\|_\infty V(\varphi).$$

Hence, it follows that

$$\left|\int_a^b f d\varphi\right| \leq \|f\|_\infty V(\varphi).$$

In particular, we see that, for each $\varphi \in BV[a,b]$, the map $F : C[a,b] \to \mathbb{F}$ defined by

$$F(f) = \int_a^b f d\varphi, \quad f \in C[a,b],$$

is a continuous linear functional on $C[a,b]$ with respect to $\|\cdot\|_\infty$, and

$$\|F\| \leq V(\varphi).$$

3 The space $NBV[a,b]$

An important subspace of $BV[a,b]$ is the space of all **normalized functions of bounded variation**, defined by

$$NBV[a,b] = \{\varphi \in BV[a,b] : \varphi(a) = 0, \text{ and } \varphi \text{ is right continuous on } [a,b)\}.$$

Thus, $\varphi \in NBV[a,b]$ if and only if $\varphi(a) = 0$ and for every $t \in [a,b)$,

$$\varphi(t+) := \lim_{h \to 0^+} \varphi(t+h) = \varphi(t).$$

It can be easily shown that

THEOREM 11. *$NBV[a,b]$ is a closed subspace of $BV[a,b]$.*

In fact, every continuous linear functional F on $C[a,b]$ can be uniquely represented by a function $\varphi_F \in NBV[a,b]$ and the map $F \mapsto \varphi_F$ is a surjective linear isometry. (cf. Limaye [2] or Nair [3]).

Exercise 12. Prove the following.

1. If μ is a Borel measure on $[a,b]$, then φ defined by $\varphi(t) = \mu([a,t])$ and $\varphi(a) = 0$ belongs to $BV[a,b]$.
2. The function φ defined by $\varphi(0) = 0$ and $\varphi(t) = t\sin(1/t)$ for $0 < t \leq 1$ is continuous but does not belong to $BV[0,1]$.
3. For $a < c < b$, the function $\varphi = \chi_{[c,b]}$ belongs to $NBV[a,b]$.

$\diamondsuit$

References

[1] N.L. Carathers, *Real Analysis*, Cambridge Univesity Press, (2000).

[2] B.V. Limaye, *Functional Analysis*, New Age International, 1996.

[3] M.T. Nair, *Functional Analysis: A First Course*, New Delhi: Printice-Hall of India, 2002 (Third Print, 2010).